<u>Μάθημα 3ο</u> <u>21/10/15</u>

Πολυωνυμική ((κ−1) διάσταση)

Πείραμα (ή δοκιμή) με κ δυνατά ασυμβίβαστα ενδεχόμενα $c_1, c_2, \ldots, c_k$ με πιθανότητα εμφάνισης: $p_i = P(c_i)$, $\sum_{i=1}^{k} p_i = 1$

n ανεξάρτητες επαναλήψεις του πειράματος.
Έστω τ.μ. X_i = αριθμός εμφάνισης (ή πραγματοποίησης) του c_i στις n επαναλήψει
Τότε, $\sum_{i=1}^{k} x_i = n$ κ' η (κ−1) διάστατη τ.μ. $(X_1, \ldots, X_{k-1})$

$$P_{\underline{X}}(x) = P_{X_1,\dots,X_{k-1}}(x_1,\dots,x_{k-1}) = P(X_1=x_1,\dots,X_k=x_k) = \binom{n}{x_1,\dots,x_k} p_1^{x_1} p_2^{x_2}\cdots p_k^{x_k} =$$

$$= \frac{n!}{x_1!\,x_2!\cdots x_k!}\, p_1^{x_1} p_2^{x_2}\cdots p_k^{x_k}$$

Έτσι: $(X_1,\dots,X_{k-1}) \sim M(n, p_1, p_2 \dots p_{k-1})$

Για $k=3$, Τριωνυμική

ΙΔΙΟΤΗΤΕΣ:

1) $X_i \sim Bin(n, p_i)$, $i=1,\dots,k-1$.

2) $(X_i, X_j) \sim M(n, p_i, p_j)$, $i,j=1,\dots,k-1$, $i\neq j$

3) $X_1 \mid X_2=x_2,\dots,X_{k-1}=x_{k-1} \sim Bin\left(n-x_2-\dots-x_{k-1}, \dfrac{p_1}{1-p_2-\dots-p_{k-1}}\right)$

Για $k=3$: ① $X_1 \sim Bin(n, p_1)$

③ $X_1 \mid X_2 \sim Bin\left(n-x_2, \dfrac{p_1}{1-p_2}\right)$

ΑΠΟΔΕΙΞΗ:

1) $(X_1, X_2) \sim M(n, p_1, p_2) \Rightarrow X_1 \sim B(n, p_1)$.

$$P_{X_1}(x_1) = \sum_{x_2=0}^{n-x_1} \frac{n!}{x_1!\,x_2!\,(n-x_1-x_2)!} p_1^{x_1} p_2^{x_2} (1-p_1-p_2)^{n-x_1-x_2} =$$

$$= \frac{n!}{x_1!(n-x_1)!} p_1^{x_1} (1-p_1)^{n-x_1} \underbrace{\sum_{x_2=0}^{n-x_1} \frac{(n-x_1)!}{x_2!\,(n-x_1-x_2)!}\, p_2^{x_2} \frac{(1-p_1-p_2)^{n-x_1-x_2}}{(1-p_1)^{n-x_1-x_2}(1-p_1)^{x_2}}}$$

$$= \sum_{x_2=0}^{n-x_1} \frac{(n-x_1)!}{x_2!\,(n-x_1-x_2)!} \left(\frac{p_2}{1-p_1}\right)^{x_2} \underbrace{\left(\frac{1-p_1-p_2}{1-p_1}\right)}_{\left(1-\frac{p_2}{1-p_1}\right)}^{n-x_1-x_2} \equiv$$

$$\equiv \sum_{x_2=0}^{n-x_1} Bin\left(n-x_1, \frac{p_2}{1-p_1}\right)$$

Οπότε τελικά $\equiv Bin(n, p_1)$

3) $P_{X_1|X_2}(x_1|x_2) = \dfrac{\dfrac{n!}{x_1!\,x_2!\,(n-x_1-x_2)!}\, p_1^{x_1} p_2^{x_2} (1-p_1-p_2)^{n-x_1-x_2}}{\dfrac{n!}{x_2!\,(n-x_2)!}\, p_2^{x_2}(1-p_2)^{n-x_2}} =$

$$= \frac{(n-x_2)!}{x_1!\,(n-x_1-x_2)!} \cdot \frac{p_1^{x_1}(1-p_1-p_2)^{n-x_1-x_2}}{(1-p_2)^{n-x_1-x_2}(1-p_2)^{x_1}} =$$

$$= \frac{(n-x_2)!}{x_1!\,(n-x_2-x_1)!} \left(\frac{p_1}{1-p_2}\right)^{x_1} \left(1-\frac{p_1}{1-p_2}\right)^{n-x_2-x_1}$$

$$\equiv Bin\left(n-x_2, \frac{p_1}{1-p_2}\right)$$

ΠΑΡΑΔΕΙΓΜΑ:

1) $X \equiv$ αριθμός {•}, $Y \equiv$ αριθμός {:}, ρίψιμο 2 (δύο) ζαριών.

$n=2$.

$p_x = \frac{1}{6}$, $p_y = \frac{1}{6}$

$(X,Y) \sim M(n=2, p_x = 1/6, p_Y = 1/6)$

$X \sim B(n=2, p_x = 1/6)$

$X|Y=0 \sim Bin\left(\underset{2-0}{n-y}, p = \frac{p_x}{1-p_Y} = \frac{1/6}{1-1/6} = \frac{1}{5}\right)$

2) $X \sim U(9,15)$

Επιθυμητή μεταξύ 10 και 12 min.

$P(X_1 = 1, X_2 = 8, X_3 = 1) = ①$

$(X_1, X_2) \sim M\left(n=10, p_1 = P(X \le 10) = \int_9^{10} \frac{1}{6} dx = \frac{1}{6}, p_2 = P(10 \le X \le 12) = \frac{2}{6}\right)$

$X \sim f(x) = \frac{1}{6}, 9 \le x \le 15.$

$① = \frac{10!}{1!\,8!\,1!} \left(\frac{1}{6}\right)^1 \cdot \left(\frac{2}{6}\right)^8 \left(\frac{3}{6}\right)^1 = 0.00114$

Υπεργεωμετρική (κ-1) διάστατη $(X_1, \dots, X_{\kappa-1}) \sim Hg(N, n, p_1, \dots, p_{\kappa-1})$
Πληθυσμός N στοιχεία σε K κατηγορίες $C_1, \dots, C_\kappa$ και έστω Np_1 είναι της $C_1, \dots, Np_\kappa$ της C_κ με $\sum_{i=1}^{\kappa} p_i = 1$ δηλ. $\sum_{i=1}^{\kappa} Np_i = N$

Χωρίς επανάθεση επιλέγονται n στοιχεία στο δείγμα και έστω x_i είναι της C_i, $i=1,\dots,\kappa$ δηλαδή $\sum_{i=1}^{\kappa} x_i = n$. Ενδιαφέρει η τ.μ. $(X_1, \dots, X_{\kappa-1})$ και συμβολίζουμε με:

$$P_{\underset{\sim}{X}}(x) = P_{X_1, \dots, X_{\kappa-1}}(x_1, \dots, x_{\kappa-1}) = P(X_1 = x_1, \dots, X_{\kappa-1} = x_{\kappa-1}) = \frac{\binom{Np_1}{x_1}\binom{Np_2}{x_2}\cdots\binom{Np_\kappa}{x_\kappa}}{\binom{N}{n}},$$

$$\sum_{i=1}^{\kappa} p_i = 1, \quad \sum_{i}^{\kappa} x_i = n.$$

ΙΔΙΟΤΗΤΕΣ:

1) $X_1 \sim Hg(N, n, p_i)$

2) $(X_1, \dots, X_{\kappa_1}) \sim Hg(N, n, p_1, \dots, p_{\kappa_1})$
$\kappa_1 < \kappa - 1.$

3) $X_1 | X_2 = x_2, \dots, X_{\kappa-1} = x_{\kappa-1}) \sim Hg\left(N - Np_2 - \dots - Np_{\kappa-1}, n - x_1 - \dots - x_{\kappa-1}, \frac{p_1}{1 - p_1 - \dots - p_{\kappa-1}}\right)$

ΑΠΟΔΕΙΞΗ:

1) $\kappa = 3$.

$$P_{X_1}(x_1) = \sum_{x_2=0}^{n-x_1} \frac{\binom{Np_1}{x_1}\binom{Np_2}{x_2}\binom{N-Np_1-Np_2}{n-x_1-x_2}}{\binom{N}{n}} = \frac{\binom{Np_1}{x_1}}{\binom{N}{n}} \underbrace{\sum_{x_2=0}^{n-x_1} \binom{Np_2}{x_1}\binom{N-Np_1-Np_2}{n-x_1-x_2}}$$

επειδή $\sum_{r=0}^{\kappa} \binom{m}{r}\binom{n}{\kappa-r} = \binom{m+n}{\kappa}$.

$$= \frac{\binom{Np_1}{x_1}\binom{\cancel{Np_2} + N - Np_1 - \cancel{Np_2}}{\cancel{x_2} + n - x_1 - \cancel{x_2}}}{\binom{N}{n}}$$

$$= \frac{\binom{Np_1}{x_1}\binom{N-Np_1}{n-x_1}}{\binom{N}{n}} \equiv Hg(N, n, p_1)$$

$$P_{X_1|X_2}(x_1|x_2) = \frac{\binom{Np_1}{x_1}\binom{Np_2}{x_2}\binom{N-Np_1-Np_2}{n-x_1-x_2}}{\binom{Np_2}{x_2}\binom{N-Np_2}{n-x_2}} = \frac{\binom{Np_1}{x_1}\binom{N-Np_1-Np_2}{n-x_1-x_2}}{\binom{N-Np_2}{n-x_2}} =$$

$$= \binom{(N-NP_2)\frac{P_1}{1-P_2}}{x_1}\binom{(N-NP_2)\left(1-\frac{P_1}{1-P_2}\right)}{n-x_2-x_1} \equiv Hg\left(N-NP_2,\ n-x_2,\ \frac{P_1}{1-P_2}\right)$$

Πολυδιάστατη κανονική

$k=2$

$$f_{X,Y}(x,y)=\frac{1}{2\pi\cdot\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}$$

$-1\leq\rho\leq 1$

$-\infty<x,y<\infty$, $-\infty<\mu_x,\mu_y<\infty$, $\sigma_x,\sigma_y>0$

$$(X,Y)\sim N\left(\begin{pmatrix}\mu_x\\ \mu_y\end{pmatrix},\begin{pmatrix}\sigma_x^2 & \rho\sigma_x\sigma_y\\ \rho\sigma_x\sigma_y & \sigma_y^2\end{pmatrix}\right),\quad \rho=\frac{Cov(X,Y)}{\sqrt{VarX\cdot VarY}}=\frac{\sigma_{xy}}{\sqrt{\sigma_x^2\sigma_y^2}}=\frac{\sigma_{xy}}{\sigma_x\cdot\sigma_y}$$

→ $\int\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy=1 \Rightarrow \frac{x-\mu_x}{\sigma_x}=u\Rightarrow dx=\sigma_x\cdot du$ και αντίστοιχα $dy=\sigma_y dv$.

$$\frac{1}{2\pi\cdot\sigma_x\sigma_y\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[u^2-2\rho uv+v^2+\rho^2v^2-\rho^2v^2\right]\right\}\sigma_x\sigma_y\,du\,dv$$

$$=\frac{1}{2\pi\sqrt{1-\rho^2}}\int\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[(u-\rho v)^2+(1-\rho^2)v^2\right]\right\}du\,dv=$$

$$=\frac{1}{2\pi\cdot\sqrt{1-\rho^2}}\int\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left[\frac{u-\rho v}{\sqrt{1-\rho^2}}\right]^2-\frac{1}{2}v^2\right\}du\,dv$$

→ $$\int\int f_{X,Y}(x,y)\,dx\,dy=\frac{1}{2\pi}\int\int_{-\infty}^{\infty}\frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2}w^2-\frac{1}{2}v^2\right\}\sqrt{1-\rho^2}\,dw\,dv$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}w^2}dw\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}v^2}=1\cdot 1=1$$

• $\frac{u-\rho v}{\sqrt{1-\rho^2}}=w\rightarrow du=\sqrt{1-\rho^2}\cdot dw$

ΙΔΙΟΤΗΤΕΣ:

1) $X\sim N(\mu_X,\sigma_X^2)$, $Y\sim N(\mu_Y,\sigma_Y^2)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \frac{1}{2\pi\,\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)v + v^2\right]\right\}\sigma_y\,dv =$$

$$\left\{\frac{y-\mu_y}{\sigma_y} = v \to dy = \sigma_y\,dv\right\}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\cdot\sigma_x}\, e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}_{N(\mu_x,\sigma_x^2)} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[v^2-2\rho v\left(\frac{x-\mu_x}{\sigma_x}\right)+\rho^2\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]\right\}dv$$

$$= N(\mu_x,\sigma_x^2)\cdot\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2} w^2\right\}\sqrt{1-\rho^2}\,dw = \qquad \left\{\frac{v-\rho\left(\frac{x-\mu_x}{\sigma_x}\right)}{\sqrt{1-\rho^2}} = w \Rightarrow dv = \sqrt{1-\rho^2}\,dw\right\}$$

$$= N(\mu_x,\sigma_x^2)\cdot\int_{-\infty}^{\infty} f(z)\,dz = N(\mu_x,\sigma_x^2)$$

$$Z \sim N(0,1)$$

2) $X \mid Y=y \sim N\left(\mu_x+\rho\frac{\sigma_x}{\sigma_y}(y-\mu_y),\ (1-\rho^2)\sigma_x^2\right)$

$Y \mid X=x \sim N\left(\mu_y+\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x),\ (1-\rho^2)\sigma_y^2\right)$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left\{\frac{1}{1-\rho^2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\rho}{1-\rho^2}(\ )(\ ) + \frac{1}{1-\rho^2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}}}{\frac{1}{\sqrt{2\pi}\cdot\sigma_y}\, e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}}$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma_y\sqrt{1-\rho^2}}\cdot e^{-\frac{1}{2}\left\{\frac{1}{1-\rho^2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2-\frac{2\rho}{1-\rho^2}(\ )(\ )+\frac{\rho^2}{1-\rho^2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}}$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left\{\frac{\frac{x-\mu_x}{\sigma_x}-\rho\frac{y-\mu_y}{\sigma_y}}{\sqrt{1-\rho^2}}\right\}^2} =$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma_y\cdot\sqrt{1-\rho^2}}\cdot e^{-\frac{1}{2}\left\{\frac{x-\left(\mu_x+\rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)\right)}{\sigma_y\sqrt{1-\rho^2}}\right\}^2} \equiv N(\ \cdot\ ,\ \cdot\ )$$

3) $\underset{\sim}{X} = (X_1,\dots,X_k)\sim N(\underset{\sim}{\mu},\Sigma)$ με $f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \frac{1}{(\sqrt{2\pi})^k|\Sigma|^{1/2}}\, e^{-\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu})\Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})}$

$\underset{\sim}{\mu} = \begin{pmatrix}\mu_1\\ \vdots\\ \mu_k\end{pmatrix}$ $\quad \Sigma = (\sigma_{ij})_{k\times k}$ $\quad \begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}$

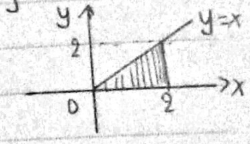
HOMEWORK

ΑΣΚΗΣΗ 2.3: Έστω X τ.μ. $X \sim U(0,1)$ και έστω η τ.μ. $Y|X=x \sim B(n, p=x)$
Να βρεθεί η κατανομή της Y.

ΑΣΚΗΣΗ 2.1: $f_{X,Y}(x,y) = \frac{1}{2} x \cdot y \, I_{(0,x)}(y) \, I_{(0,2)}(x) = \frac{1}{2} xy$ $\quad 0<x<2,\ 0<y<x$

$0<y<x<2$

Να βρεθούν οι κατανομές των X και Y (κ' η $F_{X,Y}(x,y)$)



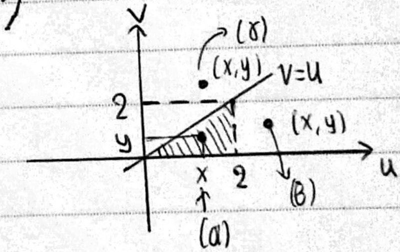
ΛΥΣΗ:

$$f_X(x) = \int_0^x \frac{1}{2} x \cdot y \, dy = \frac{x^3}{4} \quad , \quad 0<x<2.$$

$$f_Y(y) = \int_y^2 \frac{1}{2} xy \, dx = \frac{y(4-y^2)}{4} \quad , \quad 0<y<2$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f(x)} = \frac{\frac{1}{2} x \cdot y}{\frac{x^3}{4}} = \frac{2y}{x^2} \quad , \quad 0<y<x<2.$$

$$f_{Y|X=1}(y|x=1) = 2y \quad , \quad 0<y<1 \quad \left(\left(\text{Beta}(a=2, \beta=1)\right)\right)$$



$$F_{X,Y}(x,y) = 0 \quad , \quad x<0 \text{ ή } y<0$$

$$F_{X,Y}(x,y) = 1 \quad , \quad x>2 \text{ και } y>2$$

$$(a) = \int_0^y \int_v^x \frac{1}{2} u \cdot v \, du \, dv = \frac{x^2 y^2}{8} - \frac{y^4}{16} \quad , \quad 0<x<2 \text{ κ' } y<x$$

$$(\beta) = \int_0^y \int_v^2 \frac{1}{2} u \cdot v \, du \, dv = \frac{y^2}{2} - \frac{y^4}{16} \quad , \quad x>2 \ , \ y<2.$$

$$(\gamma) = \int_0^x \int_0^u \frac{1}{2} uv \, dv \, du = \frac{x^4}{16} \quad , \quad 0<x, y<2 \text{ και } y>x$$